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Optimization and Control of a Pendulum-driven Cart-pole System

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Abstract— **This paper investigates the motion generation for a pendulum-driven cart-pole system. The dynamic model of this system is developed by using the Newton's Law. A six-step motion strategy is simulated by using MATLAB/SIMULINK. The pendulum is driven by an open-loop controller to track the desired pendulum angular velocity profile. The optimal configuration which based on the profile is addressed under some specified conditions.**

Keywords- pendulum-driven; friction; propulsion mechanism; optimal configuration.

I. INTRODUCTION

Conventional wired endoscopes are commonly operated manually by a skilled operator and used to investigate the gastrointestinal tract for diagnosis of diseases. Due to the stiffness of their body, the diagnoses always generate pains to patients and cannot reach to small intestines. Capsule-type endoscope is conducted in order to avoid the inconveniences introduced by conventional endoscopes [1] [2]. However, these types of endoscopes are passive and have no propulsion mechanism. Later on, biomimetic capsule-type endoscopes which imitate the peristaltic locomotion of the inchworm are proposed [3] [4] [5] [6]. They require less complicated operation procedures and give virtually non-invasive diagnoses. Despite their advantages over conventional endoscopes, the biomimetic capsule-type endoscopes have complicated structures and require high accuracy on their temperature, position, orientation and speed.

This paper describes a new legless propulsion mechanism which can move using internal propulsive force and static friction. The idea of the mechanism is derived from a reversed experiment of the traditional inverted pendulum cart system [7], which concerning on the pendulum-driven cart using torque on the pivot. The motivation of this research is to develop a micro capsule-type robot which the patient can swallow. The micro capsule-type robot with sensors will move around in the human body and send the useful medical data to the doctors and help them to make a right decision. Similar idea was proposed by Chernousko in 2001, which was to control two-dimensional motions of a two-link mechanism along a horizontal plane by using torque apply on its hinge joint and dry friction force [8]. Similarly, to move the cart forward, a four-step motion strategy of the pendulum-driven cart system has been studied in [9]. According to [9], a full stroke of the pendulum includes high angular acceleration, low angular acceleration, low angular acceleration with the cart remains stationary and acceleration under the constraint with the cart remains stationary. In this paper, we define a six-step motion which is more accurate to implement. The open-loop control torque is calculated from the dynamic model of the system based on the desired pendulum angular velocity profile.

In section 2, we detail the development of the dynamic model of the pendulum-driven cart-pole system. In section 3, a six-step motion strategy is presented. In section 4, the optimal configuration of the average velocity of the cart is formulated and investigated. Finally, concluding remarks and future works are discussed.

II. DYNAMIC MODELING OF THE PENDULUM-DRIVEN CART-POLE SYSTEM

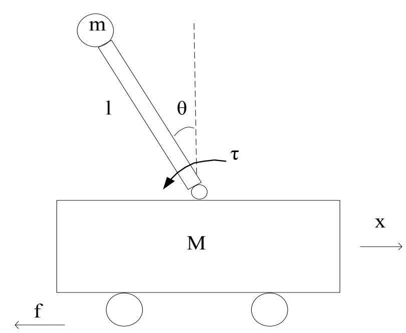

Fig.1. pendulum-driven cart-pole system

The pendulum-driven cart-pole system is shown in Fig.1. The inverted pendulum is mounted on the top of the cart. The cart has four passive wheels which make it move horizontally on the ground. A torque motor is directly attached to the pivot on the cart to swing the pendulum. M is the mass of the cart, m is the mass of the ball, l is the distance between the pivot and the mass center of the ball, μ is the friction coefficient between the cart and the ground, θ is the pendulum angle from vertical, x is the cart position coordinate, and τ is the torque applied to the pivot by the motor.

Let the center of the cart be the origin of the coordinate and the coordinate of the ball is $[x_b \; y_b]^T = [x - l\sin\theta \quad l\cos\theta]^T$. Let $F = [F_x \quad F_y]^T$ be the internal force applied to the pendulum by the torque. From the Newton's law, summing the forces applied on the ball in the horizontal direction

$$F_x = -m\ddot{x}_b$$

and summing the forces applied on the ball in the vertical direction

$$F_y - mg = m\ddot{y}_b$$

Combining above two equations, we have

$$F = \begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} -m\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta \\ mg - ml\ddot{\theta}\sin\theta - ml\dot{\theta}^2\cos\theta \end{bmatrix} \quad (1)$$

So, the torque is calculated by

$$\begin{aligned} \tau &= [l\cos\theta \quad -l\sin\theta] \cdot F \\ &= -ml\ddot{x}\cos\theta + ml^2\ddot{\theta} - mgl\sin\theta \end{aligned} \quad (2)$$

Then, summing the forces applied on the cart in the horizontal direction

$$F_x - f = M\ddot{x} \quad (3)$$

Where $f = \mu N$ is the friction force of the cart on the ground. According to the property of the friction, we select the friction coefficient as follows

$$\mu = \begin{cases} sign(F_x)\mu_k & \dot{x} = 0 \\ sign(\dot{x})\mu_k & \dot{x} \neq 0 \end{cases}$$

Summing the forces applied on the cart in the vertical direction

$$N = Mg + F_y = (M+m)g - ml\ddot{\theta}\sin\theta - ml\dot{\theta}^2\cos\theta \quad (4)$$

Putting (4) into (3) gives

$$\begin{aligned} &(M+m)\ddot{x} - ml\ddot{\theta}(\cos\theta + \mu\sin\theta) \\ &+ ml\dot{\theta}^2(\sin\theta - \mu\cos\theta) = -\mu(M+m)g \end{aligned} \quad (5)$$

Equations (2) and (5) can be written into a general compact form:

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) + N = u \quad (6)$$

where $q \in R^n$ is the generalized coordinates; $D(q) \in R^{n \times n}$ is the inertia matrix; vector $C(q,\dot{q})\dot{q} \in R^n$ represents the centripetal, Coriolis forces; $G(q)$ represents the gravitational forces; $N \in R^n$ is the external disturbances; and $u \in R^n$ is the vector of control input.

In our case,

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} x \\ \theta \end{bmatrix}, \; D(q) = \begin{bmatrix} M+m & -ml\cos q_2 \\ -ml\cos q_2 & ml^2 \end{bmatrix},$$

$$C(q,\dot{q}) = \begin{bmatrix} 0 & ml\dot{q}_2\sin q_2 \\ 0 & 0 \end{bmatrix}, \; G(q) = \begin{bmatrix} 0 \\ -mgl\sin q_2 \end{bmatrix},$$

$$N = \begin{bmatrix} \mu(M+m)g - \mu ml(\ddot{q}_2\sin q_2 + \dot{q}_2^2\cos q_2) \\ 0 \end{bmatrix}, \; u = \begin{bmatrix} 0 \\ \tau \end{bmatrix}.$$

It is noted that the dynamic equation has the following properties:

1) $D(q)$ is a symmetric positive definite matrix.

2) The matrix $[\dot{D}(q) - 2C(q,\dot{q})]$ is skew-symmetric.

3) The difference between equation (6) with the traditional cart-pole underactuated system [7] is that the input here is the torque on the pendulum joint, not the force on the cart.

III. MOTION GENERATION ANALYSIS

The six-step motion is presented as follows:

Step 1: $t \in [0, t_1)$, the motion with a high angular acceleration of the pendulum ($\ddot{\theta} >> 0$) leads to the forward accelerated motion of the cart ($\ddot{x} > 0, \dot{x} > 0$).

Step 2: $t \in [t_1, t_2)$, the motion with an angular deceleration of the pendulum ($\ddot{\theta} < 0, \dot{\theta} > 0$) leads to the forward decelerated motion of the cart ($\ddot{x} < 0, \dot{x} > 0$).

Step 3: $t \in [t_2, t_3)$, the motion with a low angular deceleration of the pendulum ($-\varepsilon < \ddot{\theta} < 0, \dot{\theta} > 0$) and the cart remains stationary ($\ddot{x} = 0, \dot{x} = 0$).

Step 4: $t \in [t_3, t_4)$, the motion with a low angular deceleration of the pendulum ($-\varepsilon < \ddot{\theta} < 0, -\delta < \dot{\theta} < 0$) and the cart remains in stationary ($\ddot{x} = 0, \dot{x} = 0$).

Step 5: $t \in [t_4, t_5)$, the motion with a zero acceleration, a low constant angular velocity of the pendulum ($\ddot{\theta} = 0, -\delta < \dot{\theta} = -w_0 < 0$) and the cart remains in stationary ($\ddot{x} = 0, \dot{x} = 0$).

Step 6: $t \in [t_5, t_6]$, the motion with a low angular acceleration of the pendulum ($0 < \ddot{\theta} < \varepsilon, -\delta < \dot{\theta} < 0$) and the cart remains in stationary ($\ddot{x} = 0, \dot{x} = 0$).

For one full stroke, the strategy can be considered as fast motion (step 1 and step 2) and slow motion (step 3, 4, 5, and 6). According to it, we define a feasible trajectory for the desired angular velocity of the pendulum $\dot{\theta}_d$. The desired plot of the pendulum angular velocity profile for one full stroke is shown in Fig.2.

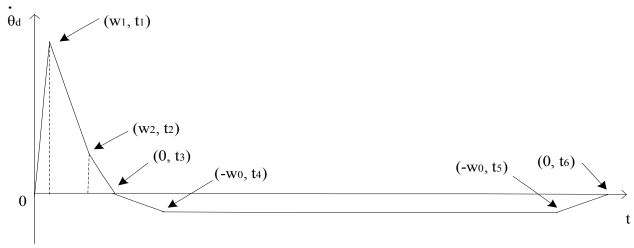

Fig.2. desired pendulum angular velocity profile

From the profile above, we define following relations

$$\frac{1}{2}w_1 t_1 + \frac{1}{2}(w_1 + w_2)(t_2 - t_1) = 2\theta_0 \quad (7)$$

$$\frac{1}{2}[(t_6 - t_3) + (t_5 - t_4)]w_0 = 2\theta_0 + \frac{1}{2}w_2(t_3 - t_2) \quad (8)$$

where $\theta_0 = -\theta(0)$, and $w_2(t_3 - t_2)/2$ is very small.

We can calculate the desired angle of the pendulum θ_d as

$$\theta_d(t) = \int_0^t \dot{\theta}_d(\sigma)d\sigma$$

and the angular acceleration of the pendulum $\ddot{\theta}_d$ as

$$\ddot{\theta}_d(t) = \frac{d\dot{\theta}_d(t)}{dt} = \frac{\dot{\theta}_d(t + \Delta t) - \dot{\theta}_d(t)}{\Delta t}$$

which Δt is infinite small.

Constraint for the whole motion: In order to keep the cart on ground, it has to satisfy the following condition

$$\begin{aligned} N &= Mg + F_y \\ &= (M + m)g - ml\ddot{\theta}\sin\theta - ml\dot{\theta}^2\cos\theta > 0 \end{aligned} \quad (9)$$

Constraint for the slow motion: During the slow motion, in order to keep the cart in stationary ($\ddot{x} = 0, \dot{x} = 0$), the internal force in the horizontal direction has to be lower than the maximal dry friction force, i.e.

$$|F_x| \le \mu(Mg + F_y) \quad (10)$$

Combining equations (1), (2) and inequality (10), we have

$$\begin{aligned} &\frac{-\mu(M + m)gl + ml^2\dot{\theta}^2(\sin\theta + \mu\cos\theta)}{(\cos\theta - \mu\sin\theta)} \\ &\le \tau + mgl\sin\theta \\ &\le \frac{\mu(M + m)gl + ml^2\dot{\theta}^2(\sin\theta - \mu\cos\theta)}{(\cos\theta + \mu\sin\theta)} \end{aligned} \quad (11)$$

Since μ is small and $\theta \in (-\pi/2, \pi/2)$, both $(\cos\theta - \mu\sin\theta)$ and $(\cos\theta + \mu\sin\theta)$ are greater than zero.

Displacement in the fast motion

During the fast motion, the cart moves forward ($\dot{x} > 0$) all the time. Using the boundary conditions:

$$\theta(0) = -\theta(t_2) = -\theta_0,\ \dot{\theta}(0) = 0,\ \dot{x}(0) = \dot{x}(t_2) = 0$$

Integrating (5) once gives

$$\begin{aligned} &(M + m)\dot{x} - ml\dot{\theta}\cos\theta - \mu ml\dot{\theta}\sin\theta \\ &= -\mu(M + m)gt \end{aligned} \quad (12)$$

Since $ml(\cos\theta_0 + \mu\sin\theta_0) \ne 0$, at time t_2,

$$w_2 = \frac{\mu(M + m)gt_2}{ml(\cos\theta_0 + \mu\sin\theta_0)} \quad (13)$$

Integrating (5) twice gives

$$\begin{aligned} &(M + m)x - ml\sin\theta + \mu ml\cos\theta \\ &= -ml(\sin\theta_0 - \mu_k\cos\theta_0) - \frac{1}{2}\mu(M + m)gt^2 \end{aligned} \quad (14)$$

Considering the boundary condition, at time t_2, we have

$$x(t_2) = \frac{2m}{M + m}l\sin\theta_0 - \frac{1}{2}\mu g{t_2}^2 \quad (15)$$

We can find that if the duration of the fast motion is shorter, the cart will move further. We can calculate the desired torque for such velocity profile using (2), and validate that the constraint (9) is satisfied. We also need to make sure the torque τ is not within the range of inequality (11).

Slow motion

In order to find a desired torque in step 3 and step 4, we can select a relative low angular deceleration of the pendulum (i.e. extending the duration of step 3 and 4) which satisfies inequality (11)

$$\ddot{\theta} = -\frac{w_2}{t_3 - t_2},\ t \in [t_2, t_3) \quad (16)$$

$$\ddot{\theta} = -\frac{w_0}{t_4 - t_3},\ t \in [t_3, t_4) \quad (17)$$

Then put it with $\ddot{x} = 0$ into (2) gives

$$\tau = \begin{cases} ml^2\ddot{\theta} - mgl\sin\theta & t \in [t_2, t_3) \\ ml^2\ddot{\theta} - mgl\sin\theta & t \in [t_3, t_4) \end{cases} \quad (18)$$

During step 5, $\dot{x} = 0, \ddot{\theta} = 0$, then

$$\tau = -mgl\sin\theta \quad t \in [t_4, t_5) \quad (19)$$

During step 6,

$$\ddot{\theta} = \frac{w_0}{t_6 - t_5},\ t \in [t_5, t_6] \quad (20)$$

So, the desired torque in step 6 is

$$\tau = ml^2\ddot{\theta} - mgl\sin\theta \quad t \in [t_5, t_6] \quad (21)$$

We can evaluate the average velocity of the cart by

$$\bar{\dot{x}} = \frac{x(t_2)}{T} = \left(\frac{2m}{M + m}l\sin\theta_0 - \frac{1}{2}\mu g{t_2}^2\right)/t_6 \quad (22)$$

Simulation results

The simulation was carried out by using MATLAB/SIMULINK. The parameters are given as $M = 0.5kg$, $m = 0.05kg$, $L = 0.3m$, $\mu_k = 0.01$, $\theta_0 = \pi/3$, $g = 9.81 m/s^2$. The simulation results for one full stroke within 0.01 second sampling interval are shown in Fig.3.

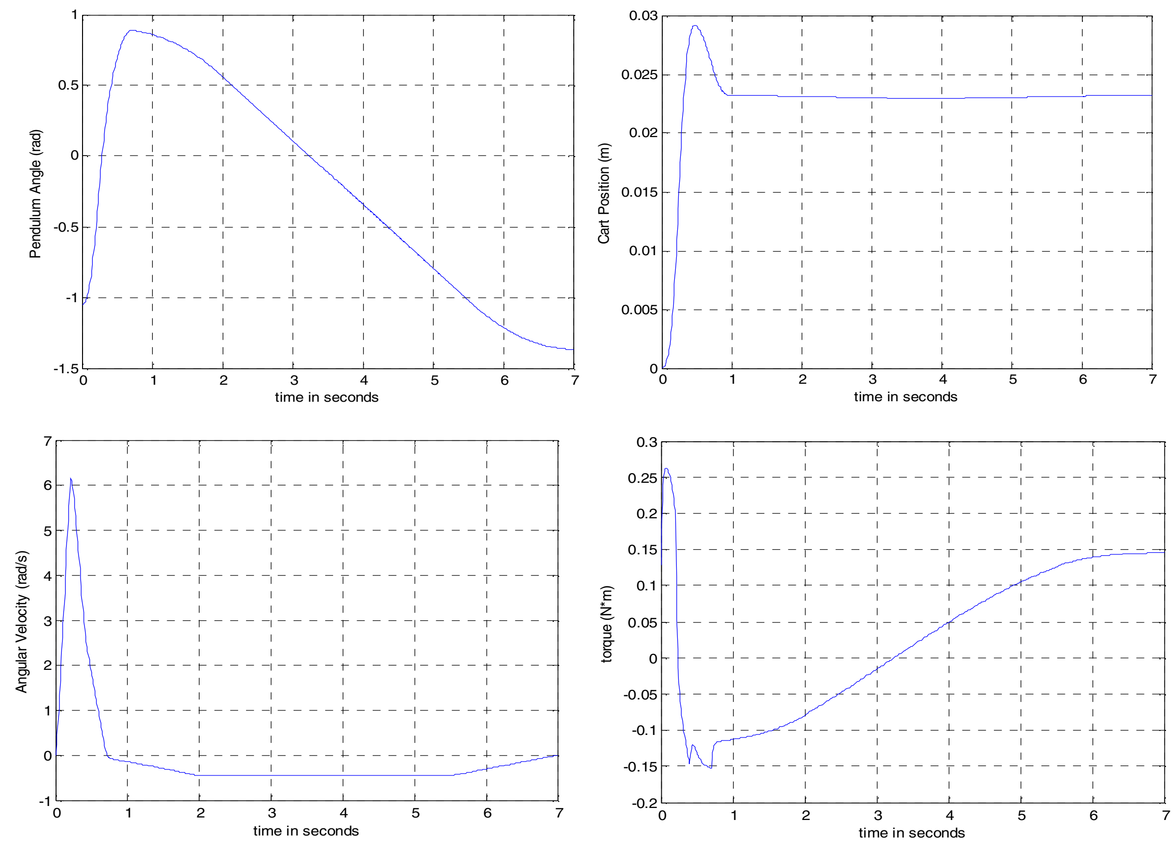

Fig.3. one full stroke of the pendulum-driven cart-pole system

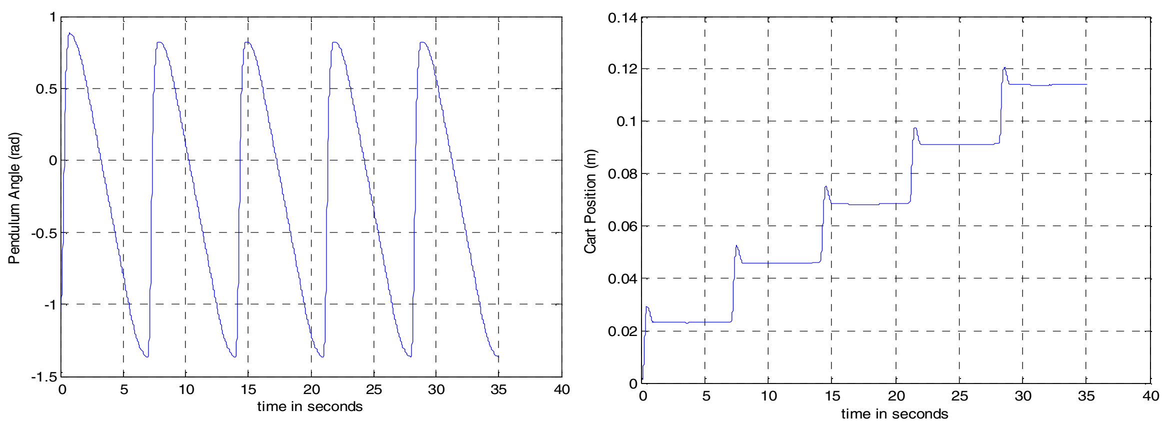

Fig.4. system trajectory in five cycles

In this simulation, we used an open-loop controller which the desired controlled torque has been calculated before the simulation. Fig.3 shows that the proposed pendulum angular velocity profile is followed properly by the simulation. Because of the sampling time, the cart velocity can not be exactly zero at time t_2. The cart moves back a short distance after step 2. Finally, the cart stops at 0.023 meters in one cycle. Fig.4 shows the system trajectory in five continuous cycles. The cart moves about 0.115 meters in five cycles which takes 35 seconds. So, the average speed of the cart is approximately 0.33 centimeter per second.

IV. OPTIMIZATION

The optimization can be specified as to maximize the average speed of the cart using the desired pendulum angular velocity profile. From fig.2, we can describe the optimization as follow:

Given $t_1 = 0.1s$, $t_2 = 0.4s$, $t_3 = 0.6s$, $t_4 = 1.4s$, $t_6 - t_5 = 0.8s$, to find $w_1, w_2, w_0, t_5 - t_4$.

Using (13), we can find

$$w_2 = 2.93 rad / s$$

Using relation (7), we have

$$w_1 = 8.27 rad / s$$

During step 5, the following equations hold

$$\ddot{x} = 0 , \ \ddot{\theta} = 0$$

So, the constraint (10) can be described as

$$\dot{\theta}^2 (|-\sin\theta| + \mu\cos\theta) \le \frac{\mu(M+m)g}{ml}$$

In our case, we can approximately consider $\theta \in [-\pi/3, \pi/3]$, the maximum value of $(|-\sin\theta| + \mu\cos\theta)$ is 0.876 . So, if we want the pendulum to swing with a constant velocity, the maximum constant velocity will be $2.03 rad / s$.

Then, using relation (8) and $w_0 = 2.03 rad / s$, we have

$$t_5 - t_4 = 0.39s$$

Here, we solve the issue of optimization. Update the desired pendulum angular velocity profile using the optimal configuration and compare the results with previous simulation.

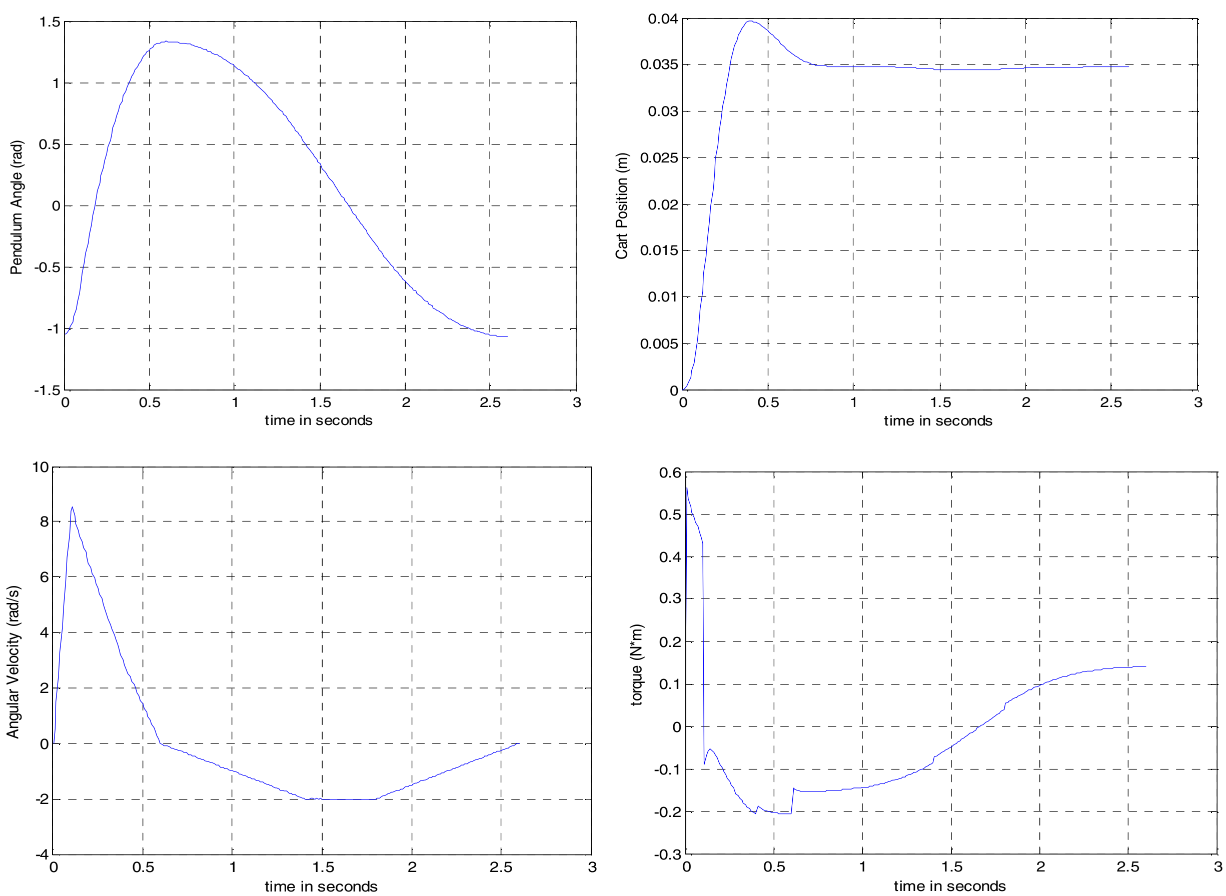

Fig.5. one full stroke of the pendulum-driven cart-pole system using optimal configuration

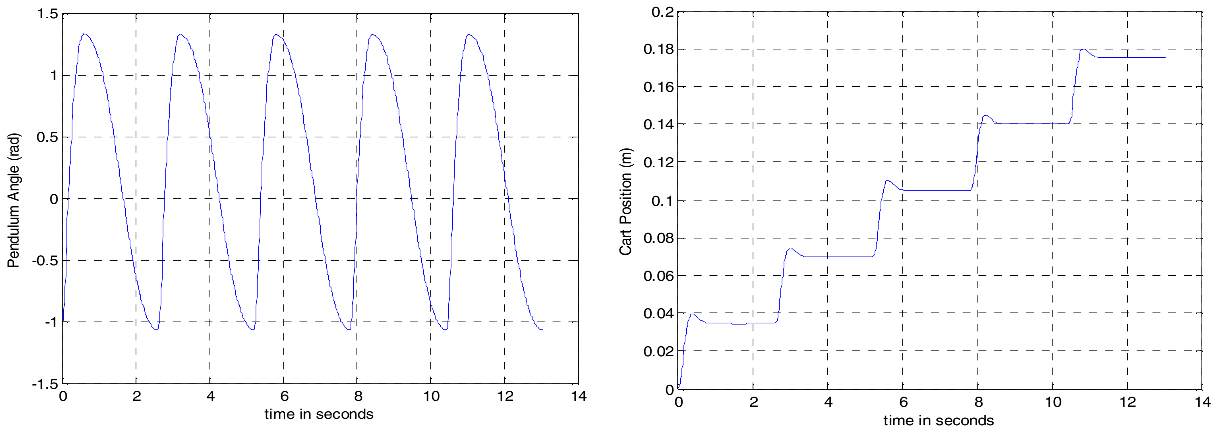

Fig.6. system trajectory using optimal configuration in five cycles

Fig.5 shows one full stroke of the system using optimal configuration. Comparing with Fig.3, the cart final position is 0.034 meter which is larger than before. The cycle time is reduced from 7 seconds to 2.6 seconds. Fig.6 shows the system trajectory using optimal configuration in five cycles. The cart finally stops at 0.17 meters which takes only 13 seconds. So, the average speed of the cart is approximately 1.31 centimeter per second.

V. Conclusions and Future Works

The dynamic model of the pendulum-driven cart-pole system is developed by using the Newton's Law. A six-step motion strategy has been analyzed in details. The calculation of the desired torque which based on the desired pendulum angular velocity profile is given and the simulation results are presented. The optimal configuration of the average speed of the cart under some specified conditions is also developed. Comparisons are discussed on the two simulation results.

The novel propulsion mechanism gives extensive applications on the micro robot, such as medical inspection robot [5], rescuing robot [10], engineering diagnosis robot, etc. Briefly, the mechanism can be used as an autonomous mobile robot to inspect the field that is inaccessible for human being. The further work along this direction is under way and the research findings will be reported in due course.

Further work includes the design of the closed-loop controller for the cart trajectory tracking, robust control with parameter uncertainties, optimization of the average velocity of the cart without specified conditions, minimal energy control of the stroke, etc.

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